Power-law-distributed level crossings define fractal behavior

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A relationship is established between the autocorrelation function of continuous Gaussian and non-Gaussian stochastic processes and the discrete process that describes their zero or level crossings. Random fractals occur when the distribution for the number of crossings is described by a class of Markov processes whose singlefold statistics are the discrete analog of the Lévy-stable continuous probability densities.

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I. INTRODUCTION

The resurgence of interest in Lévy-stable probability densities [1] has followed from recognition of their relevance to numerous instances of critical and complex phenomena (e.g., [2]). These are frequently fractal in character, being characterized by a hierarchical range of scales. However, although fractal behaviors and statistical stability frequently coexist, being different expressions of a systems' inherent complexity, the precise relationship between these concepts has not been determined. It is the purpose of this paper to establish such a connection. This requires the concept of discrete stability rather than the more familiar continuous form, and it is appropriate, therefore, to review briefly what these terms mean.

The epithet "stable" derives from the property that sums of identical and stably distributed random variables are similarly distributed and, thereby, are natural descriptors of any process that is the result of a random walk. The best-known example of this is the addition of N Gaussian random variables x, of variance σ^2, which is also described by a Gaussian distribution with variance $N\sigma^2$. A generalization of the singlefold description of a random variable leads to the Gaussian random process, which describes the evolution of $x(t)$ and is prescribed completely by the multivariate Gaussian density and autocorrelation function $\rho(\tau)=\langle x(0)x(\tau)\rangle/\sigma^2$, implying that higher-order correlations can be expressed as functions of ρ. The Gaussian random process is ubiquitous, underpinning classical statistical physics, with particular instances ranging from the description of Brownian motion (e.g., [3]) through speckle patterns associated with the random interference of coherent radiation [4], to the fluctuations of equities and currencies (e.g., [5]), among many others [2]. The class of stable distributions is broader than the Gaussian, however, encompassing variables whose probability densities $p(x)$ have power-law tails, such that $p(x)\sim 1/|x|^{1+\mu}$ with index in the range $0<\mu<2$. It follows that the variance and higher moments of these distributions do not exist.

The concept of statistical stability can be extended to encompass discrete random variables too [6,7]. Here the analog of the Gaussian distribution and its associated stochastic processes is assumed by Poisson statistics, while the scale-free distributions for the random integer $n\geqslant 0$ have the asymptotic form $P(n)\sim 1/n^{1+\nu}$ for $n\gg 1$ where the index is in the range $0<\nu<1$, so that the mean and higher moments of these distributions do not exist. Despite the mean and correlation functions being infinite, the discrete-stable distributions can be obtained as the stationary state of a recently investigated class of first-order Markov process [7], so that the evolution of the distribution from an arbitrary initial state to the stationary state is well defined and continuous in time. The case $\nu=1$ is the special case of the Poisson series of events, which is memoryless and has independent intervals between those events. This is not the case for processes whose index ν is other than unity. This paper will demonstrate the intimate connection between the discrete stable processes and the correlation function of continuous fractal Gaussian and non-Gaussian processes, and this demonstration requires exploiting properties of the zero crossings or level crossings of a process.

The average number of zero crossings of a continuous Gaussian process $x(t)$ that occur in an interval of size L depends upon the correlation function through [8]

$$\bar{n}=\frac{L}{\pi}[-\rho''(0)]^{1/2}, \tag{1}$$

and this requires $\rho(\tau)$ to be twice differentiable at the origin for $\bar{n}$ to exist or, equivalently, that $x(t)$ is continuous and once differentiable. Those correlation functions that describe an "inverse cascade" to progressively smaller scales have expansion close to the origin of the form

$$\rho(\tau)=1-a|\tau|^{2H}+\cdots, \tag{2}$$

where $0<H<1$, and consequently are not twice differentiable at the origin. They characterize self-affine fractal processes with zero crossings that cannot be resolved by magnification. Consequently, $\bar{n}$ is infinite, however small the size of the measuring interval. The Hurst exponent H [9] is a measure of the persistence of random walks whose increments are described by such a Gaussian process and is related to the fractal dimension D for the trace of $x(t)$ through $D=2-H$ [10]. Such a trace defines "fractional Brownian motion" [10]. The constant a appearing in Eq. (2) is related to ℓ, the topothesy, through

$$a=\frac{1}{2}\ell^{2(1-H)},$$

this being a measure characterizing the roughness of the fractal that is equal to the interval over which chords joining points on the fractal's trace have a root-mean-square slope of 1 radian.

It will be shown here that the zero crossings of a continuous fractal Gaussian process with Hurst exponent H are consistent with a discrete-stable series of events of index $\nu=H$. First a series of points on the "time axis" are generated by a discrete-stable process. These points are taken to correspond to the zero crossings of a random telegraph wave, whose correlation function can be easily obtained from the properties of the discrete-stable process. Using the Van Vleck theorem [11], the autocorrelation function of a Gaussian process $x(t)$ can be deduced from that of the telegraph wave, and it is shown that these have the form given by Eq. (2). The equivalence of the Hurst exponent and the index characterizing the power law of the discrete-stable distribution is thereby established. However, this result is more profound than the simple relationship between the fractal dimension and the Hurst exponent because our analysis is based upon the stochastic processes that forms this measure and it will be shown that the process formed by the zero crossings is embedded in that of the continuous variation.

The separate elements that are required to carry out this program have been developed elsewhere. In order that this paper be self-contained, those properties required of Gaussian random processes and telegraph waves will be itemized in Sec. II while the results required of the discrete-stable processes that generate the zero crossings will be given in Sec. III. Section IV contains the analysis of a Gaussian fractal process together with a calculation of the fourth-order autocorrelation function, which is shown to be consistent with the fourth-order autocorrelations of the zero crossings. Section V considers a different model for the discrete events, which is asymptotically similar to the discrete-stable distribution, and presents results that contrast the two models. The paper then extends this work to consider two classes of non-Gaussian behavior which have been selected so that they differ appreciably from the Gaussian process. The first of these, considered in Sec. VI is a continuous process with discontinuous derivatives comprising a sequence of steps of arbitrary height, which was introduced in the context of describing the scattering of radiation from "rough" corrugated surfaces [12]. Section VII considers the Γ or χ^2 process, which is strictly positive and finds a wide variety applications (e.g., [13]). The equivalent "Van Vleck theorems" for these non-Gaussian processes are required to complete the analysis. Establishing these theorems is where the technical challenges for generalizing the result principally lies, for there are comparatively few processes with which one can work analytically, and where they do exist, the form of the joint distribution is more complicated to deal with. A discussion and concluding remarks follow in Sec. VIII. Technical results concerning the determination and form of the generating functions for the series of events are consigned to Appendix A, a generalization of the results to level crossings of a Gaussian process is contained in Appendix B, and the higher-order correlation structure of the step process is contained in Appendix C.

II. SALIENT FUNDAMENTAL AND DERIVED PROPERTIES OF THE GAUSSIAN PROCESS

The Gaussian process of zero mean and unit variance is completely defined by its multivariate-density function

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{N/2}|\Lambda|^{1/2}} \exp\left(-\frac{1}{2}\mathbf{x}^T\Lambda^{-1}\mathbf{x}\right).$$

In the above, $\mathbf{x}=(x_1,x_2,\ldots,x_N)$ is a vector of real Gaussian random variables with transpose $\mathbf{x}^T$, Λ the $N\times N$ correlation matrix, whose symmetric elements $\Lambda_{ij}\equiv\rho_{ij}=\langle x_i x_j\rangle$, and $|\Lambda|$ is the determinant of the correlation matrix. The important property of the Gaussian process is that it is defined by the second-order correlation function ρ_{ij}, and consequently all higher-order correlations are functions of ρ_{ij} alone. If $\mathbf{x}$ is a stationary process that evolves in time, so that $x_1=x(t), x_2 =x(t+\tau)$, then the bivariate density is

$$p(x_1,x_2) = \frac{1}{2\pi[1-\rho(\tau)^2]} \exp\left(-\frac{x_1^2+x_2^2-2\rho(\tau)x_1x_2}{2[1-\rho(\tau)^2]^{1/2}}\right).$$

A random telegraph wave $T(t)$ can be formed by hard-limiting this process through

$$T(t) = \begin{cases} 1, & x(t)\geqslant 0,\\ -1, & x(t)<0,\end{cases}$$

and since $\langle x\rangle=0$, so too is $\langle T\rangle$. Consider the rectified telegraph wave $\theta(t)=[1+T(t)]/2$ which adopts values 0 and 1. The correlation product $\theta(t)\theta(t')$ is nonzero only if $x(t)=x_1$ and $x(t')=x_2$ are both positive. Because $x(t)$ is stationary, upon setting $x_1=x(0)$ and $x_2=x(\tau)$, the autocorrelation function of the rectified telegraph wave is simply

$$\langle\theta(0)\theta(\tau)\rangle = \int_0^\infty\int_0^\infty dx_1dx_2p(x_1,x_2),$$

whose evaluation constitutes the Van Vleck theorem [11]

$$\langle\theta(0)\theta(\tau)\rangle = \frac{1}{4}+\frac{1}{2\pi}\arcsin[\rho(\tau)] \qquad (3)$$

or, equivalently,

$$\langle T(0)T(\tau)\rangle = \frac{2}{\pi}\arcsin[\rho(\tau)]. \qquad (4)$$

Higher-order correlation functions of the rectified telegraph wave can be derived, although a closed-form expression is only possible for the triple-correlation function: viz.,

$$\langle\theta(0)\theta(\tau)\theta(\tau')\rangle = \frac{1}{8}+\frac{1}{4\pi}\arcsin[\rho(\tau)]+\frac{1}{4\pi}\arcsin[\rho(\tau')] + \frac{1}{4\pi}\arcsin[\rho(\tau-\tau')]. \qquad (5)$$

The quadruple correlation function can be shown to satisfy the relationship

$$\frac{\partial^2\langle\theta(0)\theta(\tau)\theta(\tau')\theta(\tau'')\rangle}{\partial\rho(0)\partial\rho(\tau')} = \frac{1}{(2\pi)^2|\Lambda|^{1/2}},$$

with similar expressions obtained by permuting the ρ's evaluated at different time intervals. However, it is not possible to exploit this result for anything other than the most trivial of correlation functions ρ; rather, it will prove more practical to evaluate numerically the fourfold integral

$$\langle \theta(0)\theta(\tau)\theta(\tau')\theta(\tau'')\rangle \equiv \langle \theta_1\theta_2\theta_3\theta_4\rangle = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty dx_1dx_2dx_3dx_4 \times p(x_1,x_2,x_3,x_4) \quad (6)$$

in what follows.

III. DISCRETE-STABLE PROCESS

The discrete-stable random variables [6,7,14] with probability distribution $P(n)$ are defined in terms of their generating function

$$q(s) = \sum_{n=0}^{\infty} (1-s)^n P(n) = \exp(-As^{\nu}), \quad (7)$$

where $0<\nu\leq 1$, from which the probability distribution can be obtained through differentiation:

$$P(n) = \frac{(-1)^n}{n!}\left.\frac{\partial^n q(s)}{\partial s^n}\right|_{s=1}.$$

The case $\nu=1$ corresponds to the Poisson distribution, in which case A can be identified with the mean of the distribution. For other values of ν in the specified range, the mean and higher-order moments do not exist.

The discrete-stable distributions can be obtained from the stationary state of a class of population processes, the most simple of which comprises deaths, occurring at rate μ, and multiple immigrations with rates which are particular to the variable numbers of immigrants that enter the population [7,14]. This process can be adapted to create a sequence of discrete events in time by monitoring the population, whereby "individuals" leave at rate η and n of these are counted subsequently in a time interval t. This process has a generating function

$$Q(z,t) = \sum_{n=0}^{\infty} (1-z)^n P(n,t)$$

given by [7]

$$Q(z,t) = \exp\left[-A\left(\frac{\eta z}{\bar{\mu}}[1-\exp(-\bar{\mu}t)]\right)^{\nu}\right] \times\exp\left[-A\left(\frac{\eta z}{\bar{\mu}}[1-\exp(-\bar{\mu}t)]\right)^{\nu} {}_2F_1(1,1+\nu, 2+\nu;1-\exp(-\bar{\mu}t))\right], \quad (8)$$

where $\bar{\mu}=\mu+\eta$ is the sum of the death and emigration rates of transition within and from the population and ${}_2F_1(a,b,c;z)$ is the hypergeometric function [15]. The first exponential in the above has the form of the generating function of the discrete-stable class, as can be seen by comparison with Eq. (7), while the second exponential term represents the modification due to monitoring the process.

Suppose that the n events correspond to the number of zero crossings of a telegraph wave in a time interval of duration τ, and let $P(n,\tau)$ denote the marginal distribution for these discrete events with generating function $Q(z,\tau)$. The correlation product of the telegraph wave is evidently +1 or −1 if there are, respectively, an even or odd number of zero crossings in the interval, in which case the autocorrelation function may be written as

$$\langle T(0)T(\tau)\rangle = \sum_{n=0}^{\infty} P(2n,\tau) - \sum_{n=0}^{\infty} P(2n+1,\tau) = Q(z=2,\tau). \quad (9)$$

The value of Eq. (9) for the special case $\nu=1$ is $Q(2,\tau) = \exp(-2A\eta\tau)$, which is the well-known expression for the autocorrelation function of a telegraph wave with Poisson-distributed zero crossings (e.g., [4]). In this instance $A\eta$ can be interpreted as the average zero-crossing rate.

Higher-order correlations of the telegraph wave are of interest, but require considerably more effort to determine (for special cases see [16], for example). The triple- and quadruple-order correlations will be required, for which the third- and fourth-order generating functions are needed. The details for deriving these expressions are technical and are consequently assigned to Appendix A, but the formulas that are analogous to Eq. (9) are stated here.

The triple-correlation product has value ±1 as before, but the parity depends on the initial value of T in addition to whether there are an even or odd number of crossings in two contiguous intervals of duration τ and τ'. An elementary calculation after the same fashion as that which obtained Eq. (9) yields

$$\langle T(0)T(\tau)T(\tau')\rangle = \langle T\rangle Q(z_1=2,z_2=2,\tau,\tau'), \quad (10)$$

from which it is evident that the triple-correlation function is zero for symmetric distributions. The quadruple-correlation function depends on the number of counts in three contiguous intervals and may be shown to be

$$\langle T(0)T(\tau)T(\tau')T(\tau'')\rangle \equiv \langle T_1T_2T_3T_4\rangle = Q(z_1=2,z_2=2,z_3=2,\tau,\tau',\tau''), \quad (11)$$

where the expression for $Q(z_1,z_2,z_3,\tau,\tau',\tau'')$ is derived in Appendix A.

IV. ZERO AND LEVEL CROSSINGS

Having defined the autocorrelation function for the telegraph wave from Eqs. (8) and (9), the Van Vleck theorem, encapsulated by Eq. (4), defines the autocorrelation function of a Gaussian process with identical zero crossings to be

$$\rho(\tau) = \sin\left(\frac{\pi}{2}Q(2,\tau)\right), \quad (12)$$

whereupon expanding the result for $\bar{\mu}\tau<<1$ obtains

$$\rho(\tau) \approx 1 - \frac{1}{2}\left(\frac{\pi A}{2^{1-\nu}}\right)^2 |\eta\tau|^{2\nu} + \cdots, \quad (13)$$

which has the form appropriate for a random fractal process when $0<\nu<1$ [cf. Eq. (2)], describing a cascade to small

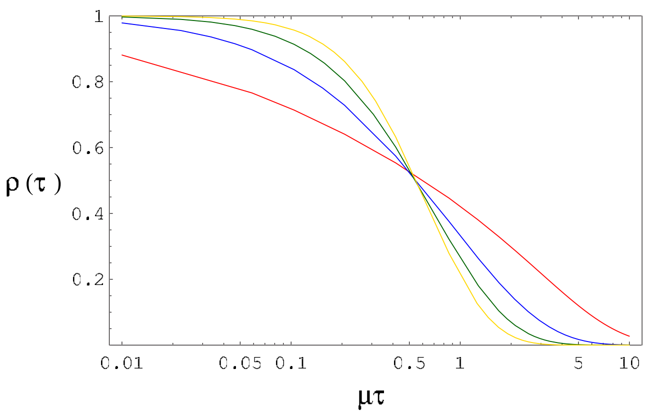

FIG. 1. (Color online) Autocorrelation function for a Gaussian process for a range of values of the index ν characterizing the power law of the zero crossings: In decreasing order from the top left corner of the figure $\nu=1/4$ (red), $\nu=1/2$ (blue), $\nu=3/4$ (green), and $\nu=1$ (yellow).

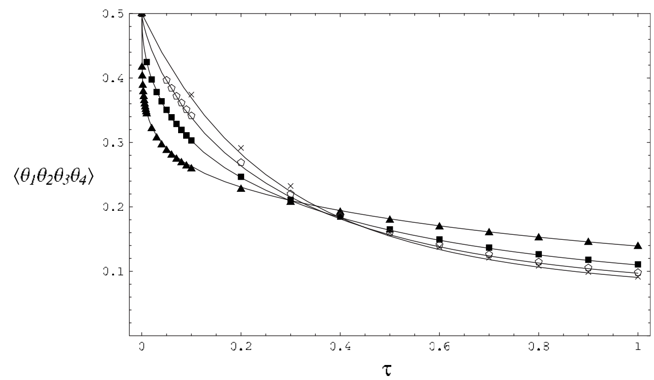

FIG. 2. Fourth-order correlation function for the rectified telegraph wave. Symbols are numerical evaluations of Eq. (6) for $\nu=1$ (crosses), 3/4, (pentagons), 1/2 (squares), and 1/4 (triangles); the solid lines are the theoretical curves determined from Eq. (14). All the time increments τ_{ij} are identical with the single delay time τ.

scales when $\bar{\mu}\tau \ll 1$. Indeed the Hurst exponent can be identified with ν the index that characterizes the discrete-stable distribution. For the special case $\nu=1$, the autocorrelation function is twice differentiable, having a Lorentzian spectrum. In this instance the use of Eq. (12) in Eq. (1) self-consistently obtains the average value for zero crossings. Because the population process is Markov in nature, the generating function is bounded exponentially,

$$Q(2,\tau) \sim \exp\left[-A\nu\left(\frac{2\eta}{\bar{\mu}}\right)^{\nu}\bar{\mu}\tau\right],$$

as $\tau\rightarrow\infty$, and so too is $\rho(\tau)$, this behavior imbuing upon the fractal a characteristic outer-scale size. Figure 1 shows the form of the Gaussian autocorrelation function for the process for a selection of values of ν.

When $\nu=H=1/2$ the continuous process is a Brownian fractal which near the origin has autocorrelation function $\rho(\tau)\approx 1-a\tau$. In this case the distribution for the number n of zeros occurring in an interval τ can be determined in closed form to be [14]

$$P(n,\tau) = \frac{2}{\pi^{1/2}n!}\left(\frac{Af(\tau)}{2}\right)^{n+1/2}K_{n-1/2}(Af(\tau)),$$

$$f(\tau) = [1-\exp(-\bar{\mu}\tau)]^{1/2}\,\text{arctanh}\{(1-\exp[-\bar{\mu}\tau])^{1/2}\},$$

where $K_n(a)$ is a modified Bessel function of the second kind [15] and this has asymptote $P(n)\sim 1/n^{3/2}$ for $n\gg 1$. The occurrence of these zeros is correlated according to

$$Q(2,\tau) = \left(\frac{1-[1-\exp(-\bar{\mu}\tau)]^{1/2}}{1+[1-\exp(-\bar{\mu}\tau)]^{1/2}}\right)^{A(\eta/2\bar{\mu})^{1/2}}.$$

The Gaussian process is uniquely determined by its autocorrelation function, given for all values of τ by Eq. (12). This implies that, for example, the fourth-order correlation function for the occurrences of the zeros must be a functional of the second-order properties. The quadruple-correlation function for the rectified telegraph wave can be determined in two ways, the purpose being to test the assertion that the process is correctly and consistently formulated. The first is through using Eq. (6), where the fourfold probability density for the Gaussian process appearing in the integrand is a function of the $\rho(\tau_{ij})=\rho(t_i-t_j)$. The left-hand side of Eq. (6) can be determined independently with the aid of Eqs. (4) and (5) to be given by

$$\begin{aligned}\langle\theta_1\theta_2\theta_3\theta_4\rangle &= \frac{1}{16}(1+\langle T_1T_2\rangle+\langle T_1T_3\rangle+\langle T_1T_4\rangle+\langle T_2T_3\rangle\\ &\quad+\langle T_2T_4\rangle+\langle T_3T_4\rangle+\langle T_1T_2T_3T_4\rangle)\\ &= \frac{1}{16}[1+Q(2,\tau_{12})+Q(2,\tau_{13})+Q(2,\tau_{14})\\ &\quad+Q(2,\tau_{23})+Q(2,\tau_{24})+Q(2,\tau_{34})+Q(2,2,2,\boldsymbol{\tau})],\end{aligned} \tag{14}$$

where $T_i\equiv T(t_i)$ and $\boldsymbol{\tau}$ denotes the dependence upon the contiguous time increments τ_{ij}. The solid curves in Fig. 2 show the quadruple correlation function as calculated using Eq. (14) with substitutions from Eqs. (A1) and (8), where all the delay times are identical. Note that the value of this function at zero-delay time is $\frac{1}{2}$. Also illustrated in Fig. 2 using symbols are values obtained following numerical evaluation of the right-hand side of Eq. (6) with autocorrelation functions substituted from Eq. (12). The agreement is to within the accuracy of the numerical quadrature and illustrates that the higher-order correlations of the Gaussian process are entirely consistent with those of the zero crossings. It should be stressed that this agreement is not a given, because the last term constituting the right-hand side of Eq. (14) is a complicated function whose form derives from the higher-order properties of the process that generates the zeros and which is not a trivial factorization of lower-order autocorrelation functions.

The association between the number of zero crossings and the autocorrelation function translates to *level* crossings too. The results differ from the zero crossing case only through the form adopted by the topothesy of the fractal, this being a function of the value of the crossing level u. Details are given in Appendix B for when u is both large and small compared with the variance of the distribution.

V. COMPARISON WITH ANOTHER DISCRETE POWER-LAW PROCESS

A pertinent question to ask is whether the discrete-stable distributions uniquely define fractal behavior or whether discrete processes that have identical asymptotics generate similar fractals. The answer to this question reveals the essential ingredient required to characterize the fractal, which is the structure of the generating function for the zero crossings at the origin.

A nonlinear population process was introduced in [17] which is a generalization of the multiple-immigration process [14]. For a critical choice of death rate μ, nonlinear coupling constant, and initial conditions, the nonlinear population process evolves to an equilibrium with generating function

$$q(s) = \frac{1}{1 + As^{\nu}},$$

where, as before, $0 < \nu < 1$. The behavior of this generating function near the origin is the same as that for the discrete-stable distributions given by Eq. (7), and so this distribution also possesses a similar power-law tail. Indeed, the distributions differ only for small values of n. The population can be monitored in the same way as was described before in order to generate a series of events, with result that the counting generating function evaluated at $z=2$ is given by

$$Q(z = 2, \tau) = \frac{H(2\eta[1 - \exp(-\bar{\mu}\tau)], 2)}{1 + A\{2\eta[1 - \exp(-\bar{\mu}\tau)]\}^{\nu}},$$

with

$$H(s,z) = \exp\left(- \int^{s} \frac{dxAx^{\nu}}{(\eta z - \bar{\mu}x)(1 + Ax^{\nu})} \right).$$

Although this has a manifestly different structure to that given by Eq. (8) which was used previously, the expansions in powers of τ near the origin are identical up to the term of order $1+2\nu$. When this Q is substituted into the Van Vleck theorem, the leading-order term will be the same as that given by Eq. (12), which defines the fractal behavior. Moreover, the discrepancy between the autocorrelation functions of the processes that are generated by the two models first occurs in the term of order $2+4\nu$. This term affects the "smooth" behavior rather than fractal characteristics of the trace. A comparison between the two models is illustrated in Fig. 3. These realizations were generated using the Fourier technique (see, for example, [18]) using an identical sequence of Gaussian random numbers of zero mean and unit variance to form the increments for the random walks. Consequently any differences between the traces are due to distinctions between the autocorrelation functions. Figure 3(a) shows part of a realization for the discrete-stable model with index $\nu=1/2$ whereas Fig. 3(b) is for the nonlinear model

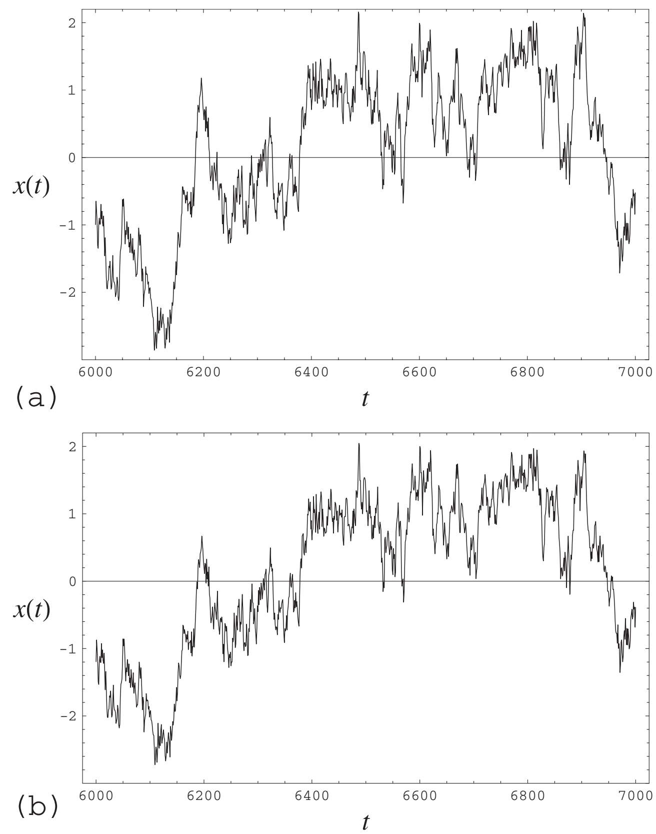

FIG. 3. Two realizations of the fractal generated by the discrete-stable model (a) and the nonlinear model (b). These both are for $\nu=1/2$, corresponding to a Brownian fractal, and are produced using identical random numbers drawn from a Gaussian distribution. The behaviors differ only in detail.

above, having identical values for all the parameters appearing in the models. Minor differences can be discerned between the two traces; for example, the cluster of zero crossings appearing near $t \sim 6550$ in Fig. 3(a) is less prominent in Fig. 3(b). Generally there are differences in the longer-scale behaviors, being consonant with where disparities between the generating functions lie. This result suggests that any generating function with expansion near the origin of the form

$$q(s) = 1 - As^{\nu} + \cdots,$$

where $0 < \nu < 1$ will, when used to form a series of events, produce a process with fractal characteristics. The index ν can then be identified with the Hurst exponent.

VI. STEP PROCESS

The step process [12] has been used to describe a corrugated surface with steps of arbitrary size x and so represents a telegraph wave with arbitrary increments having probability density function $p(x)$. The only stipulation on the density function is that the variance must exist in order that its autocorrelation function is defined at the origin. The joint distribution is given by

$$p(x_1,x_2) = p(x_2)\delta(x_1 - x_2)\rho + p(x_1)p(x_2)(1-\rho),$$

where $|\rho| \leqslant 1$ and has the generic properties that

$$\lim_{\rho\to 1} p(x_1,x_2) = p(x_2)\delta(x_1 - x_2),$$

$$\lim_{\rho\to 0} p(x_1,x_2) = p(x_1)p(x_2),$$

indicating that the process is fully correlated at zero separation and independent at large separation, respectively. The autocorrelation function of this process is given by

$$\langle x(0)x(\tau)\rangle \equiv \Lambda(\tau) = \bar{x}^2 + (\sigma^2 - \bar{x}^2)\rho(\tau),$$

where $\bar{x}$ is the mean and σ^2 the variance. Note that Λ is equal to $\sigma^2\rho(\tau)$ when the probability density function is symmetric.

The telegraph wave is defined as before and is rectified by $\theta(t)$, and both of these have coincident zero crossings with $x(t)$. The autocorrelation function of θ is given by

$$\langle \theta_1\theta_2\rangle \equiv \langle \theta(0)\theta(t)\rangle = \int_{x=0}^{\infty}\int_{x'=0}^{\infty} dx_1 dx_2 p(x_1,x_2),$$

from which it is easily shown that

$$\langle \theta(0)\theta(\tau)\rangle = [1-\wp(0)]\{1-\wp(0)[1-\rho(\tau)]\}$$

or

$$\langle T_1T_2\rangle \equiv \langle T(0)T(\tau)\rangle = 1 - 4\wp(0)[1-\wp(0)][1-\rho(\tau)], \tag{15}$$

where the positive quantity

$$\wp(x) = \int_{-\infty}^{x} dx' p(x') \leqslant 1$$

is the cumulative distribution for the step heights. If $p(x)$ is symmetric, then $\wp(0)=1/2$, in which case $\langle T_1T_2\rangle$ is identical with ρ. Equation (15) is the "Van Vleck theorem" for this process. The distribution of the zero crossings is described by the stable process with generating function given by Eq. (8), whereupon it follows that $\langle T_1T_2\rangle = Q(2,\tau) \approx 1 - A|\eta\tau|^{\nu} + \cdots$ for small values of $\eta\tau$. Inserting this into Eq. (15) one obtains

$$\rho(\tau) = 1 - \frac{1}{4\wp(0)[1-\wp(0)]}A|\eta\tau|^{\nu} + \cdots, \tag{16}$$

which is the autocorrelation function of a fractal process. A noteworthy feature is the dependence of ρ upon τ which is identical with that for the telegraph wave; this is because the process essentially *is* a telegraph wave with randomized increments. Consequently the spectrum is always in the antipersistent regime of a fractal process. When the zero crossings are Poisson distributed (i.e., $\nu=1$), then the autocorrelation function is the same as that of a Brownian fractal although, of course, the increments are not Gaussian.

If a further assumption about the step model is made—namely, that it has a single scale with higher-order statistics depending on ρ alone—then, the simple structure of the second-order joint distribution can be utilized to determine the third- and fourth-order joint distributions. This enables the fourth-order correlation function to be determined in terms of products of the autocorrelation functions, the result being given in Appendix C. The alternative is to evaluate $\langle T_1T_2T_3T_4\rangle = Q(2,2,2;\tau)$ using the third-order generating function for the series of events, as given by Eq. (C1) in Appendix C. Figure 4 compares these two functions for a symmetric distribution with $\nu=1/2$ when all the time increments are equal. The blue curve is calculated from the expression for the fourfold joint distribution, and the red curve is determined from the third-order generating function. It can be seen that these two functions are remarkably similar, there being a small discrepancy for values of $\tau \sim 0.1$; crucially, the two functions have the same behavior close to the origin, where the characteristics of the fractal behavior are determined. Once again, it is not automatic that these functions should be so similar, given the complexity and nested character of $Q(2,2,2;\tau)$ in comparison with the algebraically simple structure of the product representation in terms of the autocorrelations. Thus the single-scale model proposed in Appendix C is adequate for describing zero crossings that are described by a Markov process.

VII. Γ PROCESS

Random variables described by the Γ process are strictly positive and the analog of a zero crossing is the level crossing of some threshold. The marginal and joint probability density functions (PDFs) are given by [19]

$$p(z) = \frac{z^{\alpha-1}}{\Delta^{\alpha}\Gamma(\alpha)}\exp\left(-\frac{z}{\Delta}\right), \quad z > 0,$$

$$p(z,z') = \frac{1}{\Delta^{\alpha+1}\Gamma(\alpha)(1-\rho^2)}\left(\frac{zz'}{\rho^2}\right)^{(\alpha-1)/2} \times \exp\left(-\frac{(z+z')}{\Delta(1-\rho^2)}\right) I_{\alpha-1}\left(\frac{2\rho(zz')^{1/2}}{\Delta(1-\rho^2)}\right),$$

respectively, where $I_p(a)$ is a modified Bessel function of the first kind [15]. The Γ process has mean value $\langle z\rangle = \alpha\Delta$ and autocorrelation function $\langle z(0)z(\tau)\rangle / \langle z\rangle^2 = 1 + \rho^2(\tau)/\alpha$, which at zero delay time has the value of the normalized second moment $1+1/\alpha$. The sojourns for the Γ process exceeding a threshold u can be analyzed with the telegraph wave T, now defined as

$$T(t) = \begin{cases} 1, & z \geqslant u, \\ -1, & z < u. \end{cases}$$

The autocorrelation function for the rectified telegraph wave θ is the probability that z and z' are simultaneously above threshold—i.e.,

$$\langle \theta_1\theta_2\rangle = p(z>u) + \frac{1}{4}(\langle T(0)T(\tau)\rangle - 1) = \int_u^{\infty}\int_u^{\infty} dz dz' p(z,z'), \tag{17}$$

where $p(z>u) = \Gamma(\alpha, u/\Delta)/\Gamma(\alpha)$, with $\Gamma(\alpha,z)$ the incomplete gamma function [15]. The mean value of T is $\langle T\rangle$

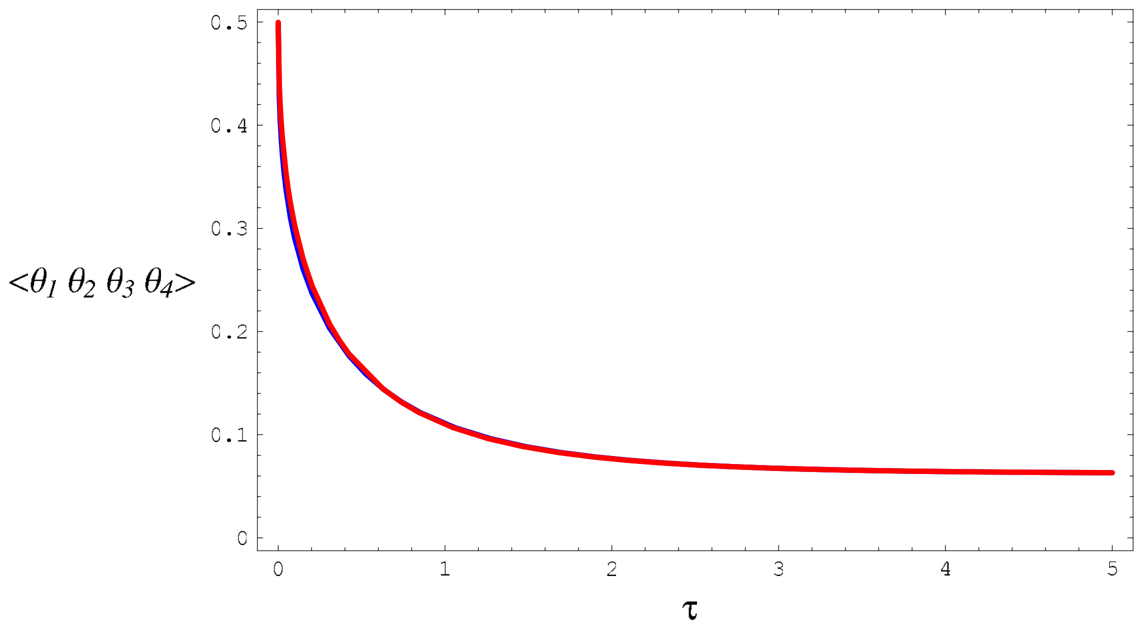

FIG. 4. (Color online) The fourth-order correlation function for the step process as calculated from the single-scale factorization model (blue) and the correlation function for the zero crossings for a stable-distributed series of events (red). The curves are practically on top of each other. The parameters for the model are $\nu=1/2$ and $\wp(0)=1/2$ with all time intervals equal. The value of the correlation function at $\tau=0$ is $1/2$, and its value as $\tau\to\infty$ is $1/16$.

$=2p(z>u)-1$. Following the method described in [19] the autocorrelation function can be evaluated exactly from

$$\langle\theta_1\theta_2\rangle=\langle\theta^2\rangle-\frac{1}{\Gamma(\alpha)}\left(\frac{u}{\Delta}\right)^{\alpha}\int_{\rho^2}^{1}d\zeta\frac{1}{(1-\zeta)\zeta^{\alpha/2}}\times\exp[-2u/\Delta(1-\zeta)]I_{\alpha}[2u\zeta^{1/2}/\Delta(1-\zeta)],$$

this being the Van Vleck theorem for the Γ process. In the above $\langle\theta^2\rangle=p(z>u)$ since $\langle T^2\rangle=1$. Only the value of the autocorrelation function near the origin is required, corresponding to $\rho^2\approx1$, in which case the modified Bessel function can be approximated by its large value asymptote through $I_p(z)\sim\exp(z)/(2\pi z)^{1/2}$ for $z\gg1$, enabling the integral to be approximated by

$$\langle\theta_1\theta_2\rangle\approx\langle\theta^2\rangle-\frac{1}{2\pi^{1/2}\Gamma(\alpha)}\left(\frac{u}{\Delta}\right)^{\alpha-1/2}\times\exp\left(-\frac{u}{\Delta}\right)\int_{\rho^2}^{1}d\zeta\frac{1}{(1-\zeta)^{1/2}\zeta^{(2\alpha+1)/4}},$$

whereupon the integral can be evaluated to yield

$$\langle\theta_1\theta_2\rangle\approx p(z>u)-\frac{1}{2\pi^{1/2}\Gamma(\alpha)}\left(\frac{u}{\Delta}\right)^{\alpha-1/2}\exp\left(-\frac{u}{\Delta}\right)\times\left(\frac{\pi^{1/2}\Gamma((3/2-\alpha)/2)}{\Gamma((5/2-\alpha)/2)}-B_{\rho^2}((3/2-\alpha)/2,1/2)\right),\tag{18}$$

with $B_\kappa(a,b)$ the incomplete beta function [15]:

$$B_\kappa(a,b)=\int_0^\kappa dt t^{a-1}(1-t)^{b-1}.$$

Supposing that the number of excursions above the threshold is governed by the discrete stable distribution enables Eqs. (8) and (9) to be utilized in the autocorrelation function for θ after which Eq. (18) can be inverted to extract ρ as a function of τ. For small values of τ, the autocorrelation function for the Γ process is given by

$$\langle z(0)z(\tau)\rangle/\langle z\rangle^2\approx1+\frac{1}{\alpha}[1-C(\alpha,u)|\eta\tau|^{2\nu}],\tag{19}$$

which is the natural extension to a non-Gaussian fractal autocorrelation function with "Hurst exponent" $H=\nu$ and can describe both persistent and antipersistent behaviors. In the above, C is a function that depends on the value of the level crossing u and the index of the Γ process α, but whose value is not critical to establishing the principals of the argument. For the apparently singular values of α appearing in Eq. (18), the integral is expressible in terms of simpler functions and the result (19) still applies.

VIII. SUMMARY AND DISCUSSION

This paper has established a fundamental connection between fractal behavior—a property governed by correlations—and discrete power-law *distributions* that describe the point statistics of a process. This connection provides a deeper, more tangible and useful association than that given by the well-known relationship for the dimension of a zero set or level set of a fractal in terms of the Hurst exponent, which informs how the variance scales with a measurement interval [10]. When the discrete-stable distributions, or distributions that are asymptotically similar to them, describe the number of level crossings that occur in an interval, then the autocorrelation function of the continuous process is consistent with that describing a fractal cascade to progressively smaller scales, and this is encapsulated in Eqs. (13), (16), and (19). It is only those distributions with power-law indices falling within the regime of the discrete-stable distributions that are able to generate fractals in this way. For the special case of a continuous process with Poisson-distributed zero crossings, the autocorrelation function is not fractal, for then the mean number of level crossings exists. The relationship has been extended to level crossings and to two examples of non-Gaussian processes. The Γ process describes a strictly positive random variable, and therefore level crossings were considered. This leads to a fractal process with Hurst exponent $H=2\nu$. It is worthy to note from Eq. (18) that there is an implicit dependence on ρ^2 that leads to the numerical factor 2 appearing in the definition of the Hurst exponent. This numerical factor is the same as for Gaussian processes, where there is also a nonlinear relationship between the autocorrelation function of the telegraph wave and that of the continuous process. Contrast this with the other non-Gaussian model considered here, the random step process whose Van Vleck theorem is Eq. (15). Note that here the dependence on ρ is linear, leading to a Hurst exponent $H=\nu$, and because $0<\nu\leqslant 1$, this describes a process that is always in the antipersistent fractal regime. While the random step process may be thought to be trivial, it has the virtue of having a simple factorisation structure when it is assumed that the process possesses a single scale. Consequently higher-order correlations can be modeled in an analytically closed form, as introduced here, and this model may be of value in other contexts.

Processes can be envisaged that are smoother in nature than those considered here. For example, there exist a class of processes that are continuous, but the *derivative* has fractal properties, and these have been termed “subfractal” (e.g., [20]). In this case the mean number of zero crossings exists because $\rho''(0)$ is finite; moreover, the second moment also exists because this is a global function of $\rho''(\tau)$, $\rho'(\tau)$, and $\rho(\tau)$. The properties of the zero crossings and extremal points for this type of process do not exhibit the simple and robust relationship described in this paper. Rather the crossings have a more complex and subtle phenomenology, their distribution belonging to one of two universality classes according to the size of the relative variance $\mathrm{Var}(n)/n$ with unity, but which are not of the discrete stable form. Moreover, their correlation properties display bunching or antibunching behaviors depending upon which of the class members they belong to. Another contrasting feature is that the distribution changes with the value of the crossing level, evolving to a Poisson distribution of pairs. This comparative richness in behavior is bound up with the size of $\rho''(0)$ relative to the next highest-order term in the expansion of the autocorrelation function. The details for this and how it connects with the current work will be presented elsewhere [21].

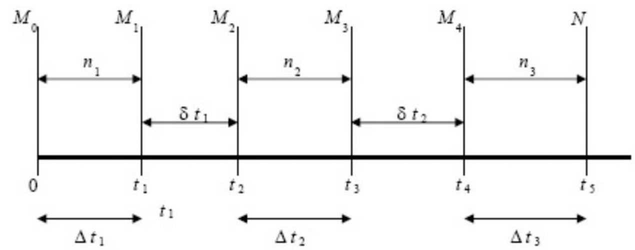

FIG. 5. The intervals required to evaluate the fourth-order generating function. The intervals during which n_j counts are registered are of duration Δt_j. These are separated by intervals of length δt_j during which the population evolves but emigrations from it are not counted. The values M_j denote the size of the monitored population at the times indicated.

ACKNOWLEDGMENT

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APPENDIX A: EVALUATION OF THE QUADRUPLE CORRELATION FUNCTION FOR THE TELEGRAPH WAVE

This appendix provides the results and methods for obtaining the higher-order generating functions defining the sequence of discrete events. The method for obtaining the generating functions generalizes that described in Ref. [22].

Figure 5 illustrates counting three successive emanations from a population. The counting procedure commences at time $t=0$ at which juncture the population has size M_0, and during the interval $(0,t_1)\equiv\tau$, n_1 counts are recorded. The population size at the end of this interval is M_1. During the subsequent interval $(t_1,t_2)\equiv\delta t_1$, no counts are recorded, but the population can still evolve, having size M_2 at the end of the period. The interval δt_1 may be interpreted as a “dead” or refractory time when the detector fails to register counts. The process proceeds with two more counting periods of duration τ' and τ'', interleaved by another dead period of length δt_2, the population at the final time t_5 being N. The joint probability for the three counts is given by

$$p(n_1,\tau;n_2,\tau';n_3,\tau'') = \sum_{\substack{M_0,M_1,M_2,\\ M_3,M_4,N}} [\mathrm{Prob}(n_1,\tau,\text{ and } M_1 \text{ at } t_1|M_0 \text{ at } t=0)\mathrm{Prob}(n_2,\tau',\text{ and } M_3 \text{ at } t_3|M_2 \text{ at } t_2)\mathrm{Prob}(M_2 \text{ at } t_2|M_1 \text{ at } t_1)$$
$$\times\mathrm{Prob}(n_3,\tau'',\text{ and } N \text{ at } t_5|M_4 \text{ at } t_4)\mathrm{Prob}(M_4 \text{ at } t_4|M_3 \text{ at } t_3)].$$

This can be converted to a generating function by forming

$$Q(z_1,\tau;z_2,\tau';z_3,\tau'') = \sum_{\substack{M_0,M_1,M_2,\\ M_3,M_4,N}} \sum_{n_1,n_2,n_3} (1-z_1)^{n_1}(1-z_2)^{n_2}(1-z_3)^{n_3}p(n_1,\tau;n_2,\tau';n_3,\tau'')$$

and noting that terms representing the conditional generating function of the form, for example,

$$\sum_{n_3,N} (1-z_3)^{n_3}\text{Prob}(n_3,\tau'' \text{ and } N \text{ at } t_5|M_4 \text{ at } t_4) \equiv \sum_{n_3,N} (1-z_3)^{n_3}(1-s)^N|\text{Prob}\,(n_3,\tau'' \text{ and } N \text{ at } t_5|M_4 \text{ at } t_4)|_{s=0}$$
$$= [1-\phi(s=0,z_3,\tau'')]^{M_4}Q(s=0,z_3,\tau''),$$

where $Q(s,z_i,\tau)$ is the joint-generating function for counting n_i events in time τ, initiated from the population having size M, and the function $\phi(s,z,t)$ describes the temporal evolution of the joint population,

$$\phi(s,z,t) = \eta z[1-\exp(-\bar{\mu}t)] + \bar{\mu}s\exp(-\bar{\mu}t).$$

The above results in summations having the structure, for example,

$$\sum_{M_4} [1-\phi(s=0,z_3,\tau'')]^{M_4}\text{Prob}(M_4 \text{ @ } t_4|M_3 \text{ @ } t_3)$$
$$\equiv (1-f\{\phi(s=0,z_3,\tau''),\delta t_2\})^{M_3}q(\phi(s=0,z_3,\tau'')),$$

where $q(s)$ is the generating function in the absence of any counting with $f\{s,t\}$ describing the evolution of the unmonitored population during the time interval t and this has the simple form $f\{s,t\}=s\exp(-\bar{\mu}t)$.

With the aid of the above two constructions, all nine summations can be evaluated analytically to obtain the generating function comprising a product of generating functions appropriate for the counted and uncounted intervals.

The quadruple correlation function for the telegraph wave can be obtained with reference to Fig. 5 by letting the unmonitored intervals δt_1 and δt_2 tend to zero, resulting in three contiguous counting intervals τ, τ', and τ'', marked by four times 0, t_1, t_3, and t_5:

$$Q(z_1,z_2,z_3;\tau,\tau',\tau'') \equiv Q(z_1,\tau;z_2,\tau';z_3,\tau'') = Q(0,z_3,\tau'')Q(f\{\phi[0,z_3,\tau''],0\},z_2,\tau')Q(f\{\phi[f\{\phi[0,z_3,\tau''],0\},z_2,\tau'],0\},z_1,\tau)$$
$$\times q(\phi[f\{\phi[f\{\phi[0,z_3,\tau''],0\},z_2,\tau'],0\},z_1,\tau]). \quad \text{(A1)}$$

Setting the $z_i=2$ obtains

$$Q(2,2,2;\tau,\tau',\tau'') = \sum_{n_1,n_2,n_3} (-1)^{n_1+n_2+n_3}p(n_1,n_2,n_3;\tau,\tau',\tau'').$$

Upon enumerating the number of sign changes that the telegraph wave $T(\tau)$ executes in three contiguous intervals, an elementary but tedious calculation shows that the above is equal to the quadruple correlation function $\langle T(0),T(\tau)T(\tau')T(\tau'')\rangle$.

APPENDIX B: LEVEL CROSSINGS OF THE GAUSSIAN PROCESS

The same technique can be used to determine the autocorrelation function of a Gaussian process through its level rather than zero crossings. Defining the rectified telegraph wave as

$$\theta(t) = \begin{cases} 1, & x(t) \geq u, \\ 0, & x(t) < u, \end{cases}$$

the autocorrelation function will be

$$\langle \theta(0)\theta(\tau)\rangle = \int_u^\infty \int_u^\infty dx\,dx'\,p(x,x'),$$

which is the Van Vleck theorem appropriate for level crossings. This has been obtained in [19] and is expressible in terms of an integral, upon which it is possible to perform appropriate asymptotic analyses depending upon the size of u relative to the standard deviation of the distribution, which here is assumed to be unity. Two limits will be examined: $u \ll 1$ and $u \gg 1$.

For $u \ll 1$, employing Eq. (27) of [19] one obtains

$$\langle \theta(0)\theta(\tau)\rangle \approx \langle\theta\rangle^2 + \frac{1}{2\pi}\sin^{-1}\rho(\tau) - \frac{u^2}{2\pi}\left[1-\left(\frac{1-\rho(\tau)}{1+\rho(\tau)}\right)^{1/2}\right] + O(u^4), \quad \text{(B1)}$$

where

$$\langle\theta\rangle = \int_u^\infty p(x)dx = \frac{1}{2}\text{erfc}\left(\frac{u}{2^{1/2}}\right) \approx \frac{1}{2} - \frac{u}{(2\pi)^{1/2}} + O(u^2).$$

Now,

$$\langle\theta(0)\theta(\tau)\rangle = \langle\theta^2\rangle - \frac{1}{4} + \frac{\langle T(0)T(\tau)\rangle}{4} = \langle\theta\rangle - \frac{1}{4}[1-Q(2,\tau)], \tag{B2}$$

the last line following from the definition of θ and use of Eq. (9) and this can be substituted into the left-hand side of (B1). Then solving (B1) in favor of $\rho(\tau)$ with the aid of the above formulas and Eq. (8) one obtains

$$\rho(\tau) \approx 1 - \frac{1}{2}\left(\frac{\pi A}{2^{1-\nu}(1-u^2/2)}\right)^2 |\eta\tau|^{2\nu} + \cdots, \tag{B3}$$

which can be seen to be a small perturbation of order u^2 on the result given by Eq. (13).

When $u \gg 1$, Eq. (29) of [19] can be utilized in the first instance: viz,

$$\langle\theta(0)\theta(\tau)\rangle = \langle\theta\rangle^2 + \frac{1}{2\pi}\exp\left(-\frac{u^2}{1+\rho(\tau)}\right)\frac{[1+\rho(\tau)]^{3/2}}{[1-\rho(\tau)]^{1/2}}\int_0^\infty d\nu\frac{\exp(-\upsilon u^2)}{(\{1+[1+\rho(\tau)]\nu\}\{1+2\nu[1+\rho(\tau)]/[1-\rho(\tau)]\})^{1/2}}, \tag{B4}$$

which may rewritten without approximation as

$$\langle\theta(0)\theta(\tau)\rangle = \left[\frac{1}{2}\operatorname{erfc}\left(\frac{u}{2^{1/2}}\right)\right]^2 + \frac{1}{2\pi}\exp\left(-\frac{u^2}{1+\rho(\tau)}\right)[1+\rho(\tau)]^{1/2}\left\{\int_0^\infty d\nu\frac{2\exp(-\nu u^2)}{\{1-\rho(\tau)+2\nu[1+\rho(\tau)]\}^{1/2}} - \int_0^\infty d\nu \exp(-\nu u^2)\frac{\{1-\rho(\tau)+2\nu[1+\rho(\tau)]\}^{1/2}}{1+[1+\rho(\tau)]\nu}\right\}.$$

The second of these integrals is of order u^{-2} relative to the first and so may be neglected. The first integral can be performed analytically to give

$$\langle\theta(0)\theta(\tau)\rangle \approx \left[\frac{1}{2}\operatorname{erfc}\left(\frac{u}{2^{1/2}}\right)\right]^2 + \frac{1}{(2\pi)^{1/2}} \times\exp\left(-\frac{u^2}{2}\right)\operatorname{erfc}\left[\left(\frac{1-\rho(\tau)}{2[1+\rho(\tau)]}\right)^{1/2}u\right].$$

This result is expanded for $1-\rho\sim 0$ such that $u<[2(1+\rho)/(1-\rho)]^{1/2}$ to give

$$\langle\theta(0)\theta(\tau)\rangle \approx \frac{1}{(2\pi)^{1/2}}\left[\frac{1}{u} - \left(\frac{1-\rho(\tau)}{\pi}\right)^{1/2}\right]\exp\left(-\frac{u^2}{2}\right) + O\left(\frac{\exp(-u^2)}{u^2}\right),$$

and the left-hand side can be evaluated for $\bar{\mu}\tau \ll 1$ with the aid of Eq. (B2) to obtain

$$\langle\theta(0)\theta(\tau)\rangle \approx \frac{1}{(2\pi)^{1/2}u}\exp\left(-\frac{u^2}{2}\right) - \frac{A}{4}|2\eta\tau|^\nu,$$

whereupon

$$\rho(\tau) \approx 1 - \exp(u^2)(\pi A)^2 2^{2\nu-3}|\eta\tau|^{2\nu} + \cdots,$$

valid for $u \gg 1$ and $\eta\tau \ll 1$, which again can be seen as a perturbation of the result given by Eq. (13). Hence there is always a regime for sufficiently small $\eta\tau$ for which the autocorrelation function is that of a fractal. The Hurst exponent does not change with the value of the level crossing, but the topothesy of the fractal does.

APPENDIX C: CORRELATION STRUCTURE OF THE STEP PROCESS

The structure of the joint characteristic function of the step process lends itself to generalizing the higher-order characteristic function for the process and hence the higher-joint probability density function. The joint-characteristic function has the simple form

$$C(\lambda_1,\lambda_2) = \rho C(\lambda_1+\lambda_2) + (1-\rho)C(\lambda_1)C(\lambda_2), \tag{C1}$$

where

$$C(\lambda) = \int_{-\infty}^{\infty} dx p(x)\exp(i\lambda x)$$

is the Fourier transform of the probability density. This structure suggests that a single-scale model with higher-order statistics depending on the single function ρ should have the following factorizations involving cyclic permutations of the Fourier variables $\{\lambda_i\}$:

$$C(\lambda_1,\lambda_2,\lambda_3) = a_{123}C(\lambda_1+\lambda_2+\lambda_3) + a_{1,2,3}C(\lambda_1)C(\lambda_2)C(\lambda_3) + \sum_{k\neq i,j}\sum_{i<j} a_{ij,k}C(\lambda_i,\lambda_j)C(\lambda_k), \tag{C2}$$

with $C(\lambda_i,\lambda_j)$ given by Eq. (C1). Equation (C2) has five terms, for which the coefficients satisfy $a_{ij,k}=a_{ji,k}$. The fourth-order characteristic function

$$C(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = a_{1234}C(\lambda_1+\lambda_2+\lambda_3+\lambda_4) + a_{1,2,3,4}C(\lambda_1)C(\lambda_2)C(\lambda_3)C(\lambda_4) + \sum_{l\neq i,j,k}\sum_{i<j<k} a_{ijk,l}C(\lambda_i,\lambda_j,\lambda_k)C(\lambda_l)$$

$$+\sum_{\substack{l,k\neq i,j\\ k<l}}\sum_{i<j} a_{ij,kl}C(\lambda_i,\lambda_j)C(\lambda_k,\lambda_l)$$
$$+\sum_{\substack{l,k\neq i,j\\ k<l}}\sum_{i<j} a_{ij,k,l}C(\lambda_i,\lambda_j)C(\lambda_k)C(\lambda_l),$$

comprising 15 terms. An elementary but tedious calculation enables the coefficients appearing in these expressions to be evaluated as functions of $\rho_{ij}\equiv\rho(|\tau_i-\tau_j|)$ through noting, for example, that

$$C(0,\lambda_2,\lambda_3)=C(\lambda_2,\lambda_3),$$

$$C(0,\lambda_2,\lambda_3,\lambda_4)=C(\lambda_2,\lambda_3,\lambda_4),$$

etc. The resulting characteristic functions can then be inverted to yield the higher-order joint-density functions, whereupon the fourth-order correlation function can be determined:

$$\begin{aligned}\langle\theta_1\theta_2\theta_3\theta_4\rangle &= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty dx_1dx_2dx_3dx_4p(x_1,x_2,x_3,x_4)\\ &= \wp(0)\{a+\wp(0)(b+c+d+e+f+g+h)\\ &\quad+[\wp(0)]^2(i+j+k+l+m+n)+e[\wp(0)]^3\},\end{aligned} \tag{C3}$$

where the coefficients

$$a+e=\frac{1}{2}(\rho_{24}\rho_{34}+\rho_{23}\rho_{24}+\rho_{23}\rho_{34}-\rho_{23}\rho_{24}\rho_{34}),$$

$$b+h+l=\frac{1}{2}\{\rho_{23}[2-\rho_{34}+\rho_{24}(\rho_{34}-1)]-\rho_{24}\rho_{34}\},$$

$$c+g+m=\frac{1}{2}\{\rho_{24}[2-\rho_{34}+\rho_{23}(\rho_{34}-1)]-\rho_{24}\rho_{34}\},$$

$$d+f+n=\frac{1}{2}[\rho_{34}(2-\rho_{24}-\rho_{23}+\rho_{24}\rho_{23})+\rho_{23}\rho_{24}\rho_{34}],$$

$$\begin{aligned}i+j+k &= (1-\rho_{23})(1-\rho_{24})(1-\rho_{34})[1-(1-\rho_{12})(1-\rho_{13})\\ &\quad\times(1-\rho_{14})],\end{aligned}$$

$$\begin{aligned}a=(1/10)(&\rho_{12}\rho_{13}\rho_{14}+\rho_{12}\rho_{13}\rho_{23}+\rho_{12}\rho_{13}\rho_{24}+\rho_{12}\rho_{13}\rho_{34}\\
&+\rho_{12}\rho_{14}\rho_{23}+\rho_{12}\rho_{14}\rho_{24}+\rho_{12}\rho_{14}\rho_{34}+\rho_{12}\rho_{23}\rho_{24}\\
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&+\rho_{12}\rho_{13}\rho_{23}\rho_{24}\rho_{34}+\rho_{12}\rho_{14}\rho_{23}\rho_{24}\rho_{34}+\rho_{13}\rho_{14}\rho_{23}\rho_{24}\rho_{34}\\
&-\rho_{12}\rho_{13}\rho_{14}\rho_{23}\rho_{24}\rho_{34}).\end{aligned}$$

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